Petroleum Engineering Handbook
Petroleum Engineering Handbook
Larry W. Lake, Editor-in-Chief

I General Engineering  John R. Fanchi, Editor
II Drilling Engineering  Robert F. Mitchell, Editor
III Facilities and Construction Engineering  Kenneth E. Arnold, Editor
IV Production Operations Engineering  Joe Dunn Clegg, Editor
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Foreword

This 2006 version of SPE’s Petroleum Engineering Handbook is the result of several years of effort by technical editors, copy editors, and authors. It is designed as a handbook rather than a basic text. As such, it will be of most benefit to those with some experience in the industry who require additional information and guidance in areas outside their areas of expertise. Authors for each of the more than 100 chapters were chosen carefully for their experience and expertise. The resulting product of their efforts represents the best current thinking on the various technical subjects covered in the Handbook.

The rate of growth in hydrocarbon extraction technology is continuing at the high level experienced in the last decades of the 20th century. As a result, any static compilation, such as this Handbook, will contain certain information that is out of date at the time of publication. However, many of the concepts and approaches presented will continue to be applicable in your studies, and, by documenting the technology in this way, it provides new professionals an insight into the many factors to be considered in assessing various aspects of a vibrant and dynamic industry.

The Handbook is a continuation of SPE’s primary mission of technology transfer. Its direct descendents are the “Frick” Handbook, published in 1952, and the “Bradley” Handbook, published in 1987. This version is different from the previous in the following ways:

• It has multiple volumes in six different technical areas with more than 100 chapters.
• There is expanded coverage in several areas such as health, safety, and environment.
• It contains entirely new coverage on Drilling Engineering and Emerging and Peripheral Technologies.
• Electronic versions are available in addition to the standard bound volumes.

This Handbook has been a monumental undertaking that is the result of many people’s efforts. I am pleased to single out the contributions of the six volume editors:

General Engineering—John R. Fanchi, Colorado School of Mines
Facilities and Construction Engineering—Kenneth E. Arnold, AMEC Paragon
Production Operations Engineering—Joe D. Clegg, Shell Oil Co., retired
Reservoir Engineering and Petrophysics—Ed Holstein, Exxon Production Co., retired
Emerging and Peripheral Technologies—Hal R. Warner, Arco Oil and Gas, retired

It is to these individuals, along with the authors, the copy editors, and the SPE staff, that accolades for this effort belong. It has been my pleasure to work with and learn from them.

—Larry W. Lake
Preface

*General Engineering*, Volume I of the new *Petroleum Engineering Handbook*, has been designed to present material that is needed by all practicing petroleum engineers. It includes chapters on mathematics, properties of fluids, rock properties, rock/fluid interactions, economics, the law, and the social context of fossil energy.

The mathematics chapters of this volume are a major departure from previous editions. The mathematical tables presented in previous editions are now readily available using hand-held calculators or software on desktop computers. The mathematics chapters present mathematical topics that petroleum engineers need to better understand the literature and the software they use on a day-to-day basis. Topics such as vibrating systems, ordinary and partial differential equations, linear algebra and matrices, and Green’s functions are introduced and references are provided for readers who would like to pursue the topics in more detail.

The discussion of fluid properties covers fluid sampling techniques; properties and correlations of oil, gas, condensate, and water; hydrocarbon phase behavior and phase diagrams for hydrocarbon systems; and the phase behavior of water/hydrocarbon systems. Two chapters consider the properties of waxes, asphaltenes, and crude oil emulsions.

Rock properties and rock/fluid interactions are discussed. The rock properties include bulk rock properties, such as porosity, elastic rock properties, and rock failure relationships. Measurement techniques and models of single-phase permeability are then presented, followed by a review of the properties that describe the interaction between rocks and fluids, notably relative permeability and capillary pressure.

In addition to mathematics, fluid properties, and rock properties, petroleum engineers need to understand economic and legal issues. Essential aspects of the economic and regulatory environment are addressed in the last section. A brief review of the role of fossil energy in the 21st century energy mix ends the volume.

I want to thank all of the authors who donated their time and expertise to the preparation of this volume.

—John R. Fanchi
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1.1 Introduction
Many mathematical tools can be used to analyze vibrational systems. One of the first mathematical tools a neophyte engineer learns is calculus. The basics of limits, differentiation, and integration permeate all of engineering mathematics. This chapter offers a cursory review of these topics and uses the mathematics of vibrations to demonstrate how the concepts operate. For more specific information on all these topics, consult relevant sections of this Handbook.

Many of the mathematical tools engineers use to evaluate and predict behavior, such as vibrations, require equations that have continuously varying terms. Often, there are many terms regarding the rate of change, or the rate of change of the rate of change, and so forth, with respect to some basis. For example, a velocity is the rate of change of distance with respect to time. Acceleration is the rate of change of the velocity, which makes it the rate of change of the rate of change of distance with respect to time. Determining the solutions to these types of equations is the basis of differential calculus.

An equation with continuously varying terms is a differential equation. If only one basis is changing, then it is an ordinary differential equation (ODE); however, if two or more bases are changing, then it is a partial differential equation (PDE). An ODE uses the notation “d” and a PDE uses $\partial$ to refer to change.

Understanding differentiation starts with an understanding of limits.

1.1.1 Limits. A graph is a useful method for determining how an equation behaves. The independent variable $t$ in Eq. 1.1 determines how the dependent variable $y$ behaves. The operators and constants in an equation specify this behavior. Fig. 1.1 shows the graph of Eq. 1.1, the distance of freefall over time with an initial velocity of zero. Down is considered negative in this equation:

$$y = \frac{-g_c t^2}{2} \quad \text{.......................... (1.1)}$$

The $x$-axis (abscissa) usually is the independent variable, and the $y$-axis (ordinate) usually is the dependent variable; however, many drilling charts hold an exception to this generality, in
that their ordinate often is the independent variable, and their abscissa is the dependent one. An example of such a drilling chart is the depth vs. time graph.

In Fig. 1.1, at the time of 3 seconds, the distance is –96.522 ft. A tangent line to the graph at 3 seconds is known as the slope ($A$) of the graph at that point. To quickly estimate the slope of the tangent, divide the rise ($\Delta y$) by the run ($\Delta t$), as shown in Eq. 1.2:

$$A = \frac{y_2 - y_1}{t_2 - t_1} = \frac{\Delta y}{\Delta t}.$$ .................................................... (1.2)

In this case, the tangent $y$ value at 2 seconds is –48.261 ft and at 4 seconds is –241.305 ft. The slope then is:

$$A = \frac{-241.305 - (-48.261)}{4 - 2} = \frac{-193.044}{2} = -96.522.$$ ................................ (1.3)

Because the units in this case are ft/sec, this slope gives the velocity at that point. It is the rate of change of the distance with respect to time.

A limit is defined as the value of a function at a given point as that point is approached from either higher or lower values (often referred to as approaching from the left or right, respectively). The limit ($Y$) of Eq. 1.1 at 3 seconds is:

$$Y = \lim_{t \to 3} \frac{-g t^2}{2},$$ .................................................... (1.4)

$Y$ is known as the limit of the function. In this simple case, $Y$ is the same regardless of whether $t$ approaches 3 from the left or the right. This is not true in all cases, however (e.g., with a discontinuous function). In these cases, the limit can be determined analytically. One can also determine the limit using a graph such as in Fig. 1.1.

Limits have the following properties:

If $\lim_{t \to z} f(t) = P$ and $\lim_{t \to z} g(t) = Q$, then $\lim_{t \to z} [f(t) + g(t)] = P + Q$, .................... (1.5)

$$\lim_{t \to z} rf(t) = r \lim_{t \to z} f(t) = rP,$$ .................................................... (1.6)
\[
\lim_{{t \to z}} \frac{f(t)}{g(t)} = \frac{P}{Q}, \quad \text{as long as } Q \neq 0, \quad \text{........................................... (1.7)}
\]

and
\[
\lim_{{t \to z}} \sum_{j=0}^{J} r_j t^J - j = \sum_{j=0}^{J} r_j z^J - j. \quad \text{........................................... (1.8)}
\]

1.1.2 Derivatives. As noted earlier, the slope of graph of Eq. 1.1 at 3 seconds = \(-96.522\) ft/sec and is the velocity (\(v\)) of free-fall at 3 seconds from release. This value is known as the first derivative of Eq. 1.1 at the value of 3. It is written as:
\[
f'(t) = \frac{d}{dt} f(t) = \frac{d}{dt} y = v = -96.522. \quad \text{........................................ (1.9)}
\]

and is defined as:
\[
\frac{dy}{dt} = \lim_{{\Delta t \to 0}} \frac{f(t + \Delta t) - f(t)}{\Delta t}. \quad \text{.................................................. (1.10)}
\]

As the limit of the value of \(\Delta t\) approaches zero, the solution converges to the first derivative.

Derivatives have the following properties (\(r = \text{constant}\)).

If \(y = f(t) = r\), then \(\frac{dy}{dt} = f'(t) = 0. \quad \text{........................................... (1.11)}\)

If \(y = f(t) = rt^n\), then \(\frac{dy}{dt} = f'(t) = rnt^{n-1}. \quad \text{........................................... (1.12)}\)

If \(y = f(t) = \sin (rt)\), then \(\frac{dy}{dt} = f'(t) = r \cos (rt). \quad \text{........................................... (1.13)}\)

If \(y = f(t) = \cos (rt)\), then \(\frac{dy}{dt} = f'(t) = -r \sin (rt). \quad \text{........................................... (1.14)}\)

If \(y = f(t) = e^{rt}\), then \(\frac{dy}{dt} = f'(t) = re^{rt}. \quad \text{........................................... (1.15)}\)

If \(y = f(t) = \ln (rt)\), then \(\frac{dy}{dt} = f'(t) = \frac{1}{t}. \quad \text{........................................... (1.16)}\)

In the case of Eq. 1.7, where \(Q = 0\), L’Hopital’s rule can help find the limit. This is shown in Eq. 1.17:
\[
\lim_{{t \to z^+}} \frac{f(t)}{g(t)} = \lim_{{t \to z^+}} \frac{f'(t)}{g'(t)}, \quad \text{where } f(t) = g(t) = 0
\]

or where \(f(t) = g(t) = \pm \infty. \quad \text{........................................... (1.17)}\)
Other rules regarding differentials are the following.

The linear superposition rule:

\[
\text{If } f(t) = \sum_{j=1}^{J} f_j(t), \text{ then } f'(t) = \sum_{j=1}^{J} f'_j(t). \tag{1.18}
\]

The product rule:

\[
\frac{d}{dt}[f(t)g(t)] = f(t) \frac{d}{dt}g(t) + g(t) \frac{d}{dt}f(t). \tag{1.19}
\]

The quotient rule:

\[
\frac{d}{dt}\left[\frac{f(t)}{g(t)}\right] = \frac{1}{g(t)^2} \frac{d}{dt}f(t) - \frac{f(t)}{g(t)^2} \frac{d}{dt}g(t). \tag{1.20}
\]

The chain rule (or function of a function):

\[
\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}. \tag{1.21}
\]

Multiple differentiations can be shown by

\[
\frac{d^2y}{dt^2} = \frac{d^2}{dt^2}f(t) = f''(t), \tag{1.22}
\]

and continued differentiations can be shown by

\[
\left\{\frac{d}{dt}\left[\frac{dy}{dt}\frac{dy}{dt}f(t)\right]\right\} = \frac{d^3}{dt^3}f(t) = f'''(t). \tag{1.23}
\]

A useful point to recognize is where a slope equals zero, which can correspond to a maximum, a minimum, or an inflection. To determine these points, determine a first derivative of an equation. Then, set this first-derivative equation to equal zero and solve for the basis (the unknown). To determine whether this point is a maximum, a minimum, or an inflection, determine the second derivative of that equation. If that value is negative, the point is a maximum; if it is positive, the point is a minimum; and if it is zero, the point is an inflection.

The graph of Eq. 1.24 (Fig. 1.2) is an example of this process:

\[
y = t^3 - 5t^2 + 3t + 5. \tag{1.24}
\]

The first derivative of Eq. 1.24 is:

\[
\frac{dy}{dt} = 3t^2 - 10t + 3, \tag{1.25}
\]
which, when set equal to zero, is a quadratic equation with two roots, \( t = 3 \) and \( 1/3 \). These two points correspond to the maximum and minimum points on the graph. To prove which is which, a second derivative is taken:

\[
\frac{d^2y}{dt^2} = 6t - 10, \quad \text{................................. (1.26)}
\]

which at \( t = 3 \) and \( 1/3 \) is equal to 8 and –8, respectively. This means that at \( t = 3 \), the function is at a minimum and at \( t = 1/3 \), the function is at a maximum.

The first differentiation of the equation of the position of a free-falling object starting at rest (Eq. 1.1) gives the slope of the graph, which, as noted, is the velocity:

\[
\frac{dy}{dt} = v = g_c t, \quad \text{................................. (1.27)}
\]

A second differentiation gives the change of the slope with respect to time (acceleration), and is:

\[
\frac{d^2y}{dt^2} = g_c, \quad \text{................................. (1.28)}
\]

which is the acceleration caused by Earth’s gravity.

1.1.3 Differential-Equation Solutions. Solutions to differential equations solved in closed form can range from trivial to impossible. Numerical methods often are required. Nevertheless, some general strategies have been developed to solve differential equations.\(^{1,2,3}\)

An ODE with only first derivatives is known as a first-order ODE. A second-order ODE has second and possibly first derivatives. The same reasoning applies to third order and beyond. Likewise, when a PDE has only first derivatives, it is a first-order PDE. The second and third orders and beyond are defined on the basis of their highest-order derivative.

This section has covered some of the basics of ODE and PDE mathematics. The reader is urged to review mathematical texts and handbooks for more details on this subject.

To illustrate some of the aforementioned principles of ODE and PDE mathematics, the rest of this chapter discusses examples within the subject of vibrations. Vibration mathematics has
been chosen because vibrations are notorious for instigating drilling and production problems. The next section covers some basic information on the nature of vibrations, with subsequent sections devoted to some aspect of the mathematical models of vibrations.

1.2 Introductory Thoughts About Vibrations
The fundamental theories of vibration are not new. Indeed, Saint-Venant published his theory on the vibrations of rods in 1867, and Love published an entire treatise on vibration theory in 1926. The mathematics of vibration theory involves infinite series, complex functions, and Fourier integral transforms, and its physics involves Newtonian mechanics and stress analyses. Until recently, except under relatively simple conditions, the complexity of such mathematics had restrained the application of vibration theory to solving simple common problems. Now, however, state-of-the-art computers can perform these complex calculations in a reasonable time frame, making possible a wave of new studies.

1.3 Vibration Theory
A vibration is a fluctuating motion about an equilibrium state. There are two types of vibration: deterministic and random. A deterministic vibration is one that can be characterized precisely, whereas a random vibration only can be analyzed statistically. The vibration generated by a pumping unit is an example of a deterministic vibration, and an intermittent sticking problem within the same system is a random vibration.

In mechanical systems, deterministic vibrations are excitations that elicit a response from a system, as shown schematically in Fig. 1.3. In theory, as long as two of the three variables (excitation, system, and response) are known, the third one can be determined; however, the mathematics might be challenging. Most often, the response function is sought, so that the excitation function and the system must be known.

Vibration systems can be linear or nonlinear, and discrete or continuous (Fig. 1.4). In all cases, a vibration system can be in one, two, or three mutually orthogonal dimensions. A linear system is a system in which proportionality (Eq. 1.29) and superposition (Eq. 1.30) are true, that is, in which:

If \( E(t) \rightarrow R(t) \), then \( rE(t) \rightarrow rR(t) \) ........................................... (1.29)

and

If \( E_1(t) \rightarrow R_1(t) \) and \( E_2(t) \rightarrow R_2(t) \),
then \( E_1(t) + E_2(t) \rightarrow R_1(t) + R_2(t) \). ........................................... (1.30)

When proportionality and superposition are not true, then the system is nonlinear.

A discrete system is one having a finite number of independent coordinates that can describe a system response. These independent coordinates are known as degrees of freedom (DOFs). If the motion of mass, either translational or rotational, of a vibrating system is a function of only one independent coordinate, then the system has one DOF. If two or more independent coordinates are required to describe one or both types of motion, then the system has two or more DOFs. If a system is continuous (an infinite set of independent coordinates is
needed to describe the system response), it has an infinite number of DOFs. Because material structures all have a continuous nature, all systems have an infinite number of DOFs. Most systems have dominant DOFs; some even have a single dominant DOF. Such systems therefore can be characterized as discrete systems, which makes the mathematics more tractable.

If a system has a single DOF or set of DOFs in only one direction, it is a 1D system. If there are two mutually orthogonal directions for the DOF, it is a 2D system; and if there are three mutually orthogonal directions for the DOF, it is a 3D system.

As Fig. 1.5 shows, the excitation function can be periodic or transient, and absent or present. A periodic vibration is one that can be characterized mathematically as an indefinite repetition. A transient vibration is of finite length and is composed of waves that have a definite beginning and that eventually die out. These waves can be of extremely short duration or last for some time.

A standing wave is a vibration whose wave profile appears to be standing still, though actually the particles that make up the material are oscillating about an equilibrium position. Because of the geometry and boundary conditions of the material through which they are traveling, the waves and the reflected waves cancel and reinforce themselves over the same location in the material, which makes the wave profile appear not to be moving. The point at which no motion is occurring is a nodal point, or node. The point of maximum amplitude is the antinode.

In reality, all waves are transient in some way. If a wave is repeated over a longer time than it takes for a single wave to propagate through a material, then this series of waves can be called a vibration. All vibrations are transient, as well. If the vibration lasts longer than the time under analysis, then it can be characterized as infinite in length.
When the excitation is present and is actively affecting the system within the analysis time frame, the response is called a forced vibration. The response of a system with an absent excitation function—one that is not present within the analysis time frame—is called a free vibration. As such, the system can be responding to the removal of an excitation function. For example, if the response of a mass and spring system is sought after the system has been pulled down and released, the original excitation function (the pulling force) is considered absent because the analysis is being performed after the release.

1.4 Wave Propagation
The method by which a vibration travels through a system is known as wave propagation. When an external force is impressed on a real-world elastic body, the body does not react instantly over its entire length. The point immediately under the external force reacts first, and then the section just under that point reacts to the previous section’s reaction, and so on. This series of reactions is called wave propagation because the reactions propagate through the body over a period of time at a specific velocity. If the rate of change of the external force is slow enough, static equilibrium analysis can model the reactions adequately for most engineering applications. This is called rigid-body analysis. If the external force changes rapidly, however, wave-propagation analysis is necessary to model the reactions effectively.

1.4.1 Types of Waves. There are many types of elastic waves. Some listed in this section are longitudinal, lateral, and bending waves. Some of these are shown in Fig. 1.6.

In longitudinal waves (also variously called compression/tension, axial, dilatational, and irrotational waves), the particles that make up the elastic medium are forced directly toward and away from each other, and the direction of the particles’ motion is parallel to that of the wave motion. In most steels, longitudinal waves travel at ≈16,800 ft/sec. Longitudinal waves are not dispersive. This means that all the wave components that make up a longitudinal wave travel at the same velocity and, hence, do not separate (disperse).

In lateral waves (also known variously as shear, torsional, transverse, equivoluminal, and distortional waves), the particles slip beside each other, and move perpendicular to the direction of the wave motion. Because slipping uses more energy, lateral waves are slower. In steel, for example, they travel at 10,400 ft/sec. A rapidly changing torsional force on a section of
pipe will cause a lateral wave to propagate from the point of application to all other parts of
the pipe. It propagates as an angular twist. Lateral waves are nondispersive and have a similar
solution method as the longitudinal waves; however, shear or transverse waves are dispersive
(i.e., the wave components that make up a shear wave travel at different velocities). Their
wave components will disperse and “smear” the initial wave profile. This complicates the anal-
ysis significantly.

Bending waves (flexural waves) travel as a bend in a bar or plate and have longitudinal
and lateral components. Rotary shears and moments of inertia complicate bending-wave analysis.

Wave-propagation studies in petroleum engineering areas generally have been confined to
longitudinal, torsional, and lateral waves. Of these, longitudinal waves generally are easiest to
model and are considered in this section. A compression wave is a stress wave in which the
propagated stress is in compression. Likewise, a tension wave is a stress wave in which the
propagated stress is in tension.

1.5 Wave Behavior
Wave velocity depends primarily on density and modulus of elasticity but also is affected by
damping and frequency. For example, hitting one end of a long steel rod with a hammer will
generate a longitudinal wave that compresses the particles of the steel. The wave’s length is set
by the length of time that the hammer is in contact with the end of the rod, whereas its magni-
tude is set by the force of the hammer blow. As the wave moves along the rod, the steel
within the length of the wave is compressed. After the wave passes, the steel returns to its
unstressed state, though not necessarily in the same location as before the wave passage.

As another example, twisting (shearing) a steel rod will generate a shear wave. A shear
wave moves along the rod more slowly than the longitudinal wave does. Similarly to the longi-
tudinal wave discussed above, its length is set by the duration of the twisting action, whereas
its magnitude is set by the torque from the twisting action.

Waves act independently, but the stresses they create can be additive. For example, two
equal compression waves that are generated simultaneously by hammer blows at each end of a
long steel rod will meet in the center of the rod, pass through one another, and then each con-
tinue along the rod as if the other never existed (independence). While the waves are passing
each other, however, the compression in the steel will be twice (additive) that of either wave.

1.5.1 Natural Frequencies and Resonance. Everything has a natural frequency, a frequency
at which it would vibrate were it given the energy to vibrate and left alone. For instance, the
human body has a natural frequency of ≈5 cycles/sec. All drill and rod strings have a natural
frequency that depends on the material properties and geometry. The material properties deter-
mine the wave velocity, and the geometry determines how waves are reflected and refracted.

During wave propagation, the wave eventually reaches an end of the material. Some of the
wave will reflect back to its source. If the reflection reaches the source at the same time a new
wave is generated, the two waves will combine and be synchronized in phase. Later, if those
two waves’ reflections return to the source at the same time the next new wave is generated,
all three waves will combine. This will continue for as long as waves are generated under
these conditions, and the resultant wave will increase in amplitude, theoretically to infinity.
This is called resonance. The frequency at which resonance occurs is the natural frequency or
an integer multiple of that frequency (called a harmonic). If this wave reinforcement is allowed
to continue, the system eventually will either self-destruct or fatigue to failure.

A continuous system contains an infinite number of natural frequencies, whereas a discrete
single-degree-of-freedom (SDOF) system (e.g., a point mass on a massless spring) has only
one natural frequency. If two point masses are connected using two springs, then there are two
natural frequencies in this 2DOF system. In general, the number of DOFs in a system deter-
mines the number of natural frequencies it has, which means that any discrete system will have
a finite number of natural frequencies; however, in reality, there is an infinite number of natural frequencies because all systems are continuous. Some frequencies will have higher amplitudes than others. Such continuous systems with discrete higher-amplitude responses can be modeled with a discrete methodology.

1.5.2 Damping. Resonance energy does not reach an infinite value because of damping, the dissipation of energy over time or distance.\(^7\) Without damping, or friction, the energy from vibrations would build until there is more energy than the structure can sustain, which can cause structural failure.

A wave propagating into a system adds energy to a system, whereas damping removes it. Generally, the dissipated energy from the vibration is converted to heat, and if damping does not take enough energy out of a system, the system can self-destruct from energy overload. The amount of energy in a system at a given time is reflected in the system’s stress/strain level. The more stresses/strains in the system, the higher the energy level. Once the stresses reach a value greater than the yield strength of the system, yield failure is imminent. If the stresses are greater than the ultimate strength of the material, failure is immediate.

In the borehole, three distinctive types of damping occur: viscous, Coulomb, and hysteretic. Viscous damping occurs when the damping force generated is proportional to the velocity of the particles. Coulomb damping (also called dry friction) is the force generated by the movement of materials past one another, and it usually is proportional to the force normal to the materials’ surfaces. The dynamic and static coefficients of friction are the proportionality constants. Hysteretic damping is the friction force generated by the relative motion of the internal planes of a material as a wave causes particle motion. Although this is true of all materials, some materials are viscoelastic (i.e., they show a much larger hysteretic effect than do others).

**Viscous Damping.** As noted above, viscous damping occurs when the damping force is proportional to the velocity of the particles. Viscous damping is shown by:

\[
F_d = cy' \frac{dx}{dt}, \quad \text{.......................... (1.31)}
\]

One way that viscous damping arises in jarring analysis is from the interaction of a solid and liquid at their interface, such as where the steel contacts the liquid mud along the sides of a drillstring.

One method for determining the damping involves noting the decrement of acceleration over one vibration cycle.\(^8\) An impulse is impressed on the drillstring to produce a wave. While the wave is decaying, the acceleration is measured and recorded multiple times at one location on the string and at the same phase (i.e., crest to crest). The time between recordings also is noted. These values are used in Eq. 1.32 to compute the damping coefficient (\(c\)). Unfortunately, though, this method gives the total damping and does not distinguish between viscous and Coulomb damping.

\[
c = 2 \frac{A \frac{E}{v_s^2t}}{\ln \frac{a_1}{a_2}}, \quad \text{........................................... (1.32)}
\]

**Coulomb Damping.** Coulomb damping is the friction that occurs when two dry surfaces slide over each other, and its force is a constant value that is independent of particle velocity and displacement, but dependent on the friction factor (\(\mu\)) and the force normal to the friction surface. This value is:

\[
F_f = \mu F_n, \quad \text{........................................... (1.33)}
\]
The Coulomb damping force always is of the opposite sign from that of the particle velocity, so that the damping force reverses when the particle velocity changes signs. This discontinuity makes it a nonlinear damping force, shown as:

\[ F_f = \pm \left( \frac{dx}{dt} \right)_n \mu F_n. \] ................................. (1.34)

Nonlinearity makes a closed-form solution to an equation of motion difficult.

**Hysteretic Damping.** Hysteretic damping also is called structural damping because it arises from internal friction within a structure. A wave moves through a material because the atomic structure is reacting to an applied force. As the atoms of the structure move, energy is lost through the interaction of these atoms with their neighboring atoms. Hysteretic damping is the energy lost when atoms move relative to each other.

If a material had a perfectly linear stress/strain relationship, hysteretic damping would not occur. In reality, though, there is no such thing as a perfectly linear stress/strain curve. Two curves develop on the stress/strain diagram while a material is stressed and relieved. The center area between these two curves represents the energy lost to internal friction. (This hysteresis loop is the reason for the name of this damping type.) This variation can be small, but the amount of energy dissipated can be large because high-frequency vibrations can cause this loop to be repeated many, many times over a given time period.\(^9\)

The hysteretic-damping value is highly dependent on a number of factors. One factor is the condition of the material (i.e., chemical composition, inhomogeneities, and property changes caused by thermal and stress histories). Another is the state of internal stress from initial and subsequent thermal and stress histories. Also, the type and variation of stress—axial, torsional, shear, and/or bending—affect the hysteretic-damping value.

A way of looking at hysteretic-damping force is to set it proportional to the particle velocity divided by the wave frequency. This is shown in Eq. 1.35.\(^{10}\)

\[ F_h = \frac{h}{\omega} \frac{dx}{dt}. \] ............................................................... (1.35)

### 1.6 Equivalent Springs

Many systems can be modeled as multiple springs. Such springs can be combined into a single, equivalent spring (Fig. 1.7). For parallel springs, the sum of the spring constants is equal to the equivalent spring constant (Eq. 1.36). For series springs, the reciprocal of the sum of the reciprocals of the spring constants is equal to the equivalent spring constant (Eq. 1.37). A linear spring oscillates in a single translational direction. A torsional spring oscillates with an angular twist (Eq. 1.38).

### 1.7 Boundary and Initial Conditions

The boundary conditions (how the ends of a system are attached) and initial condition (condition of the system at the start in time) are extremely important in vibration and wave propagation analysis. The specific solution of any ODE or PDE requires a set of boundary and/or initial conditions. Usually, a displacement (boundary condition) and an initial velocity (initial condition) are specified.

In wave propagation, the boundary conditions also dictate wave behavior. For example, a compression wave is reflected from a free end as a tension wave and from a fixed end as a compression wave. If two rods are connected at their ends and are of different geometry or material, then a fraction of the energy of the wave is reflected and the remaining portion of the energy is refracted at their connection. Other types of boundaries direct the system response by
limiting the DOF. This includes boundary conditions of pinned, revolute, translational, translational and rotational; forcing function; mass spring and/or damper, and a semi-infinite connection. In addition, changes in material properties will affect the various constants and will cause wave-propagation reflections and refractions at the boundary between the properties. Fig. 1.8 shows some typical boundary conditions.

### 1.8 Mechanical Vibration Analysis

There are three components to mechanical vibration analysis: to determine the geometric compatibilities; to determine the constitutive (material properties) equations; and to determine the equilibrium condition.

The geometric compatibilities are the displacement constraints and connections. They also include the continuous properties, which state that the system does not separate into individual components.
pieces. (If it does, that is another problem altogether.) The constitutive equations represent the material properties, which include mass, damping, and spring coefficients. These constitutive equations include stress/strain relationships and Hooke’s law (Eqs. 1.39a and 1.39b):

\[ \sigma = E \varepsilon \] \hspace{1cm} (1.39a)

or, in another form,

\[ F = \frac{E A_c}{L} \Delta l \] \hspace{1cm} (1.39b)

The coefficient of \( \Delta l \) in Eq. 1.39b often is called the spring constant or stiffness constant.

The equilibrium condition is based on both static and dynamic conditions. A static equilibrium states that the sum of the forces acting on an object is equal to zero:

\[ \sum F = 0. \] \hspace{1cm} (1.40)

A dynamic equilibrium is based on Newton’s second law and is the basis of many vibration analysis methods. The sum of the forces acting on an object is equal to its mass times the acceleration of the object. Other dynamic-equilibrium analysis includes virtual work methods and energy-balance methods (Hamilton’s principle).

Newton’s second law for a translational system is

\[ \sum F = m \ddot{x} \] \hspace{1cm} (1.41)

and for torsional systems is

\[ \sum T = I \ddot{\theta} \] \hspace{1cm} (1.42)

Newton’s second law can be rewritten in a form known as D’Alembert’s principle:

\[ \sum F - m \ddot{x} = 0, \] \hspace{1cm} (1.43)

in which \( m \ddot{x} \) is treated as a force and is called an inertial force.

Some basic equations of vibration analysis are shown in Table 1.1.

\section*{1.9 SDOF System}

An SDOF system is the most basic vibration analysis. The typical SDOF system is the classical linear oscillator (CLO), as shown in Fig. 1.9. It consists of a point mass, spring, and damper.

\subsection*{1.9.1 Free Vibration Without Damping.}

The first analysis is free vibration without damping. Using Newton’s second law and D’Alembert’s principle, the equation that describes free vibration without damping \((c = 0)\) is:

\[ \sum F = k x, \] \hspace{1cm} (1.52)

which gives \( k x = m \ddot{x} \), which when rearranged becomes:

\[ m \ddot{x} + k x = 0. \] \hspace{1cm} (1.53)
The solution to this differential equation is:

\[ x = C_1 \sin \sqrt{\frac{k}{m}} t + C_2 \cos \sqrt{\frac{k}{m}} t \] ............................................... (1.54)

The constant multiplying the \( t \) is the natural frequency of the system and is:

\[ \sqrt{\frac{k}{m}} = \omega_n \] ............................................... (1.55)
in radians/unit time. Multiplying by $\frac{1}{2\pi}$ gives the natural frequency in cycles/unit time. When Eq. 1.55 is substituted into Eq. 1.54, the result is:

$$x = C_1 \sin \omega_n t + C_2 \cos \omega_n t. \quad \text{(1.56)}$$

The constants $C_1$ and $C_2$ are based on the initial and boundary conditions. If at time 0, $x = x_0$ and $\dot{x} = v_0$, the initial location and velocity, respectively, the first coefficient is:

$$C_2 = x_0, \quad \text{................................. (1.57)}$$

which is the initial location. Differentiating once gives:

$$\dot{x} = C_1 \omega_n \cos \omega_n t - C_2 \omega_n \sin \omega_n t, \quad \text{................................. (1.58)}$$

which, when $t = 0$, gives the other coefficient:

$$C_1 = \frac{v_0}{\omega_n}, \quad \text{................................. (1.59)}$$

which is based on the initial velocity. The entire equation then is:

$$x = x_0 \cos (\omega_n t) + \frac{v_0}{\omega_n} \sin (\omega_n t). \quad \text{................................. (1.60)}$$

**Ex. 1.1** is an SDOF free and undamped CLO system.

### 1.9.2 Free Vibration With Damping

The second analysis of free vibration is with damping (Fig. 1.10). Using Newton’s second law, the equation that describes free vibration with damping ($c \neq 0$) is:

$$\sum F = kx + cx, \quad \text{................................. (1.61)}$$

which is rearranged as before to get:

$$m \ddot{x} + c \dot{x} + kx = 0. \quad \text{................................. (1.62)}$$

The general solution to this differential equation is:

$$x = C_1 e^{-c + (\sqrt{c^2 - 4mk})t} + C_2 e^{-c - (\sqrt{c^2 - 4mk})t}, \quad \text{................................. (1.63)}$$

although the specific solution depends on the value under the square root. When $c^2 - 4mk = 0$, the system is critically damped. Another way to look at this critical damping point is:

$$c_{\text{crit}} = \sqrt{4mk} = 2m\omega_n, \quad \text{................................. (1.64)}$$
Often, the damping coefficient is divided by the critical damping coefficient to get the critical damping ratio:

\[ \frac{C}{2m\omega_n} = \frac{C}{c_{\text{crit}}} = \xi \] ................................. (1.65)

If \( \xi > 1 \), the system is underdamped. When disturbed, the system will experience an oscillating decay. If \( \xi < 1 \), the system is overdamped and, when disturbed, will die out without oscillating. If \( \xi = 1 \), the system is critically damped and also will not oscillate.

A useful simplifying equation is the “damped” natural frequency, \( \omega_d \). It is:

\[ \omega_d = \omega_n \sqrt{1 - \xi^2} \] ................................. (1.66)

If the system is underdamped, that is, if \( 0 \leq \xi < 1 \), the solution is:

\[ x = e^{-\xi \omega_n t} \left( C_1 \cos \omega_d t + C_2 \sin \omega_d t \right) \] ................................. (1.67)

or

```
Example 1.1—SDOF free and undamped CLO system.
```

\[ x = x_0 \cos(\omega_n t) + \frac{v_0}{\omega_n} \sin(\omega_n t) \] ................................. (1.60)

where \( m = 2 \text{ lbm}, k = 0.5 \text{ lbf/ft}, \omega_n = \sqrt{\frac{k}{m}} = 2.836 \text{ Hz}, x_0 = 1 \text{ ft}, \) and \( v_0 = 0 \text{ ft/sec}. \]
$x = \left( \sqrt{C_1^2 + C_2^2} \right) e^{-\zeta \omega_n t} \sin \left( \omega_d t + \phi \right), \quad \ldots \ldots \ldots \ldots \quad (1.68)$

where $\phi = \tan^{-1} \left( \frac{C_1}{C_2} \right)$. If the initial and boundary coefficients are the same as before, then the solution is:

$x = e^{-\zeta \omega_n t} \left[ x_0 \cos \omega_d t + \frac{v_0 + \zeta \omega_n x_0}{\omega_d} \sin \omega_d t \right], \quad \ldots \ldots \ldots \ldots \quad (1.69)$

or

$x = \sqrt{x_0^2 + \left( \frac{v_0 + \zeta \omega_n x_0}{\omega_d} \right)^2} e^{-\zeta \omega_n t} \sin \left( \omega_d t + \tan^{-1} \left( \frac{x_0 \omega_d}{v_0 + \zeta \omega_n x_0} \right) \right), \quad \ldots \ldots \ldots \ldots \quad (1.70)$

**Ex. 1.2** is an SDOF free and underdamped CLO system.

If the system is overdamped, that is, if $\xi > 1$, the solution is:

$x = e^{-\xi \omega_n t} \left( C_1 \cosh \omega_d t + C_2 \sinh \omega_d t \right), \quad \ldots \ldots \ldots \ldots \quad (1.71)$

but in this case, $\omega_d = \omega_n \sqrt{\xi^2 - 1}$. The order changed because it was an imaginary number. With the same initial and boundary conditions as before, the solution is as before:

$x = e^{-\xi \omega_n t} \left( x_0 \cosh \omega_d t + \frac{v_0 + \zeta \omega_n x_0}{\omega_d} \sinh \omega_d t \right), \quad \ldots \ldots \ldots \ldots \quad (1.72)$

**Ex. 1.3** is an SDOF free and overdamped CLO system.

If the system is critically damped, that is, if $\xi = 1$, the solution with the initial and boundary conditions is:
\[ x = e^{-\omega_n t} \left( x_0 \cos \omega_n t + \frac{v_0 + c \omega_n x_0}{\omega_n} \sin \omega_n t \right) \]  \hspace{1cm} (1.69)

where \( m = 2 \text{ lbm}; k = 0.5 \text{ lbft/ft}; \omega_n = \sqrt{\frac{k}{m}} = 2.836 \text{ Hz}; c = 0.025 \text{ lbft/sec}; \)

\[ c_{\text{cr}} = 2m\omega_n; \quad \xi = \frac{c}{c_{\text{cr}}} = 0.07; \quad x_0 = 1 \text{ ft}; \quad \omega_n = \omega_n \sqrt{1-\xi^2} = 2.829 \text{ Hz}; \quad v_0 = 0 \text{ ft/sec}. \]

Example 1.2—SDOF free and underdamped CLO system.

\[ x = e^{-\omega_n t} \left[ x_0 + \left( v_0 + \omega_n x_0 \right)t \right] \]  \hspace{1cm} (1.73)

**Ex. 1.4** is an SDOF free and critically damped CLO system.

1.9.3 **Forced Vibration Without Damping.** The next sets of systems have a forcing function driving the vibration. The first of these is a CLO without damping (\( c = 0 \)), as shown in Fig. 1.11. The equation of motion for this system with \( F = F_0 \sin \omega_f t \), a sinusoidally varying force, using Newton’s second law, is:

\[ m \ddot{x} + kx = F_0 \sin \omega_f t \]  \hspace{1cm} (1.74)

In this case, there are two terms in the solution, the homogenous or transient term, and the particular or steady-state term. The homogenous term is the same as in a free-vibration case and is solved by setting the forcing function to zero (that is, the free-vibration case, Eq. 1.60). If the same initial and boundary conditions are applied as before, the solution for the homogenous case is the same as before:
The second term is the effect of the forcing function on the system. This is solved by assuming a particular solution and deriving it back:

\[ x_p = C_1 \sin \omega_f t + C_2 \cos \omega_f t. \] ............................................ (1.76)

\[ \dot{x}_p = C_1 \omega_f \cos \omega_f t - C_2 \omega_f \sin \omega_f t. \] ........................................... (1.77)

\[ \ddot{x}_p = -\omega_f^2 C_1 \sin \omega_f t - \omega_f^2 C_2 \cos \omega_f t. \] ....................................... (1.78)

Substituting the above equations into Eq. 1.74 gives:

\[ m(-\omega_f^2 C_1 \sin \omega_f t - \omega_f^2 C_2 \cos \omega_f t) \]
\[ x(t) = e^{-\alpha t} \left[ x_0 + (v_0 + \omega_n x_0) t \right], \] ...

\[ = F_0 \sin \omega_f t. \] ........................... (1.79)

Collecting the terms gives:

\[ \left( k - m \omega_f^2 \right) C_1 \sin \omega_f t + \left( k - m \omega_f^2 \right) C_2 \cos \omega_f t = F_0 \sin \omega_f t. \] ........................... (1.80)

Equating coefficients shows that:

\[ C_1 = \frac{F_0}{k - m \omega_f^2} \] .................................................. (1.81)

and

\[ C_2 = 0. \] .................................................. (1.82)

Therefore, the particular solution is:

\[ x_p = \frac{F_0}{k - m \omega_f^2} \sin \omega_f t, \] .................................................. (1.83)
which can be rewritten as:

\[ x_p = \frac{F_0}{k} \left( \frac{1}{\omega_f^2 - \omega_n^2} \right) \sin \omega_f t, \quad \text{..................................................} \quad (1.84) \]

where the reciprocal term in the parentheses sometimes is called the magnification factor. The total solution is:

\[ x = x_h + x_p = x_0 \cos (\omega_n t) + \left( \frac{v_0}{\omega_n} \right) \sin (\omega_n t) \]

\[ + \frac{F_0}{k} \left( \frac{1}{\omega_f^2 - \omega_n^2} \right) \sin (\omega_f t), \quad \text{..................................................} \quad (1.85) \]

Ex. 1.5 is an SDOF forced and undamped CLO system.

Note that when the forcing frequency, \( \omega_f \), matches the natural frequency, \( \omega_n \), the value of the coefficient is infinity. This is the resonance condition, and it can lead to excessively large displacements (see Fig. 1.12).

When the forcing frequency is close to but not at the natural frequency, a beating phenomenon occurs. This appears as a low frequency impressed over the frequency of the system. When the engines of a twin-engine aircraft are not quite synchronized, for example, one can hear a beating sound as a low-frequency pulse (the “wow-wow” throb). Fig. 1.13 illustrates this beating phenomenon.

1.9.4 Forced Vibration With Damping. The second system with a forcing function driving the vibration is a CLO with damping \( (c \neq 0) \), as shown in Fig. 1.14. The equation of motion for this system with the same force as before, \( F = F_0 \sin \omega t \), is:
\[ m\ddot{x} + c\dot{x} + kx = F_0 \sin \omega_f t. \] ................................. (1.86)

The solution has two parts, as before. It is similar to the last example, except for an additional damping term. The particular solution is solved similarly to the last example in Eqs. 1.76, 1.77, and 1.78. Differentiating and substituting into Eq. 1.86 gives:

\[ m\left(-\omega_f^2 C_1 \sin \omega_f t - \omega_f^2 C_2 \cos \omega_f t\right) + c\left(C_1 \omega_f \cos \omega_f t - C_2 \omega_f \sin \omega_f t\right) + k\left(C_1 \sin \omega_f t + C_2 \cos \omega_f t\right) = F_0 \sin \omega_f t. \] ................................. (1.87)

Rearranging gives:

\[ \left[(k - m\omega_f^2)C_1 - c\omega_f C_2\right] \sin \omega_f t + \left[c\omega_f C_1 + (k - m\omega_f^2)C_2\right] \cos \omega_f t = F_0 \sin \omega_f t. \] ................................. (1.88)

Equating coefficients as before yields:

\[ (k - m\omega_f^2)C_1 - c\omega_f C_2 = F_0. \] ................................. (1.89)
\[ c \omega f C_1 + \left( k - m \omega f^2 \right) C_2 = 0. \] ................................. (1.90)

Solving for the constants gives:

\[ C_1 = \frac{k \left( 1 - \frac{\omega f^2}{\omega_n^2} \right) F_0}{\left[ k \left( 1 - \frac{\omega f^2}{\omega_n^2} \right)^2 + c^2 \omega f^2 \right]} \] ................................. (1.91)
and

\[ C_2 = \frac{-c\omega_f F_0}{k \left(1 - \frac{\omega_f^2}{\omega_n^2}\right)^2 + c^2 \omega_f^2}, \] ................................. (1.92)

which gives the particular solution:

\[
x_p = F_0 \left\{ \frac{k \left(1 - \frac{\omega_f^2}{\omega_n^2}\right)}{\left[k \left(1 - \frac{\omega_f^2}{\omega_n^2}\right)^2 + c^2 \omega_f^2\right]} \sin \omega_f t \right.
\]

\[
+ \left[ k \left(1 - \frac{\omega_f^2}{\omega_n^2}\right)^2 + c^2 \omega_f^2 \right] \cos \omega_f t \right\}, ................................. (1.93)
\]

The total solution is homogenous and the particular solutions added together. In this case, if the same initial and boundary conditions are applied as before, the homogenous solution is the same as in the free-vibration case and is Eq. 1.69. The particular solution is Eq. 1.93 and is the effect of the forcing function on the system. As noted before, the critical damping coefficient dictates the behavior of the homogenous part of the solution.

**Ex. 1.6** is an SDOF forced and underdamped CLO system.

### 1.10 Two-Degree-of-Freedom (2DOF) Systems

At the basic level, systems with two and more DOFs are similar to SDOF systems. In a 2DOF system, two independent equations of motion are required to define a system (e.g., a double CLO or a double pendulum), but the DOFs need not be the same. For example, a system could have a translational and a rotational DOF. As long as the equations of motion are independent of one other, it is a 2DOF system. These equations must be solved simultaneously.
Example 1.6—SDOF forced and underdamped CLO system.

\[ x_h(t) = e^{-\xi \omega_n t} \left( x_0 \cos(\omega_d t) + \frac{v_0 + \xi \omega_n x_0}{\omega_d} \sin(\omega_d t) \right) \]  
\[ x_p(t) = F_0 \left[ \frac{k \left( 1 - \frac{\omega_0^2}{\omega_n^2} \right)}{\left( k \left( 1 - \frac{\omega_0^2}{\omega_n^2} \right) \right)^2 + \left( k \left( 1 - \frac{\omega_0^2}{\omega_n^2} \right) \right)^2} \sin(\omega_d t) + \frac{-c \omega_d}{\left( k \left( 1 - \frac{\omega_0^2}{\omega_n^2} \right) \right)^2 + \left( k \left( 1 - \frac{\omega_0^2}{\omega_n^2} \right) \right)^2} \cos(\omega_d t) \right] \]

and

\[ x = x_h + x_p, \]

in which \( m = 2 \) lbm; \( k = 0.5 \) lbft/ft; \( \omega_n = \sqrt{\frac{k}{m}} = 2.836 \) Hz; \( c = 0.025 \) lbft/sec; \( c_{\text{eq}} = 2m\omega_n; \)

\( \xi = \frac{c}{c_{\text{eq}}} = 0.07; \) \( x_0 = 1 \) ft; \( \omega_0 = \omega_n \sqrt{1 - \xi^2} = 2.829 \) Hz; \( v_0 = 0 \) ft/sec; \( F_0 = 1 \) lb; and \( \omega_d = 1 \) Hz.
1.10.1 Free Undamped 2DOF System. As in the SDOF system, an undamped \((c_1 \text{ and } c_2 = 0)\) system will be developed first. Fig. 1.15 shows such a system. As before, Newton’s second law can determine the equations of motion. In the SDOF system, a solution in the form of sine and cosine was used. For the first 2DOF system, another valid form of a solution, a sine with a phase angle, \(\Phi\), is used to show another solution form. The equation for mass 1 is

\[
m_1 \ddot{x}_1 = -k_1 x_1 + k_2 (x_2 - x_1) \tag{1.94}
\]

and for mass 2 is

\[
m_2 \ddot{x}_2 = -k_2 (x_2 - x_1) \tag{1.95}
\]

These two equations of motion must be solved simultaneously because they are coupled through the displacement terms, \(x_1\) and \(x_2\). As before, a solution is assumed and substituted back into the equations of motion. The assumed solution is

\[
x_1 = C_1 \sin (\omega t + \Phi) \tag{1.96}
\]

and

\[
x_2 = C_2 \sin (\omega t + \Phi), \tag{1.97}
\]

for which the second differentiation is
\[ x_1 = -\omega^2 C_1 \sin (\omega t + \Phi) \] .................................................... (1.98)

and

\[ x_2 = -\omega^2 C_2 \sin (\omega t + \Phi) \] .................................................... (1.99)

Substituting back into the equations of motion and collecting terms, the result is:

\[ (k_1 + k_2 - \omega^2 m_1)C_1 - k_2 C_2 = 0 \] ................................................ (1.100)

and

\[ -k_2 C_1 + (k_2 - \omega^2 m_2)C_2 = 0 \] ................................................... (1.101)

The only way not to have a trivial solution \( (C_1 = 0 \text{ and } C_2 = 0) \) to these equations is to have the determinate of the coefficients be zero:

\[ \begin{vmatrix} k_1 + k_2 - \omega^2 m_1 & -k_2 \\ -k_2 & k_2 - \omega^2 m_2 \end{vmatrix} = 0 \] ........................................... (1.102)

Using linear algebra, the determinate is:

\[ m_1 m_2 \omega^4 - [m_1 k_2 + m_2 (k_1 + k_2)] \omega^2 + k_1 k_2 = 0, \] ........................................... (1.103)

which is a quadratic equation in terms of the square of the natural frequencies, \( \omega^2 \). This solves to:

\[ \omega^2 = \frac{m_1 k_2 + m_2 (k_1 + k_2) \pm \sqrt{[m_1 k_2 + m_2 (k_1 + k_2)]^2 - 4 m_1 m_2 k_1 k_2}}{2 m_1 m_2} \] .................................................. (1.104)

There are two roots to this equation, which means that there are two natural frequencies. The solution therefore must be in terms of the two frequencies. Substituting the first natural frequency back into the equation of motion solution gives the result of the first natural frequency:

\[ (k_1 + k_2 - \omega_1^2 m_1)C_{11} - k_2 C_{21} = 0 \] ................................................ (1.105)

and

\[ -k_2 C_{11} + (k_2 - \omega_1^2 m_2)C_{21} = 0 \] ................................................... (1.106)

The ratio of \( C_{11} \) to \( C_{21} \), known as the mode shape, for the first natural frequency is:
\[ \lambda_1 = \frac{C_{11}}{C_{21}} = \frac{k_2}{k_1 + k_2 + \omega_1^2 m_1} = \frac{k_2 - \omega_2^2 m_2}{k_2} \]  \tag{1.107}

therefore,

\[ C_{11} = \lambda_1 C_{12} \] \tag{1.108}

The first natural frequency motions then are:

\[ x_1 = C_{11} \sin (\omega_1 t + \Phi_1) = \lambda_1 C_{21} \sin (\omega_1 t + \Phi_1) \] \tag{1.109}

and

\[ x_2 = C_{21} \sin (\omega_1 t + \Phi_1) \] \tag{1.110}

The mode shape for the second natural frequency is:

\[ \lambda_2 = \frac{C_{12}}{C_{22}} = \frac{k_2}{k_1 + k_2 + \omega_2^2 m_1} = \frac{k_2 - \omega_2^2 m_2}{k_2} \] \tag{1.111}

therefore,

\[ C_{12} = \lambda_2 C_{22} \] \tag{1.112}

The second natural frequency motions then are:

\[ x_1 = C_{12} \sin (\omega_2 t + \Phi_2) = \lambda_2 C_{22} \sin (\omega_2 t + \Phi_2) \] \tag{1.113}

and

\[ x_2 = C_{22} \sin (\omega_2 t + \Phi_2) \] \tag{1.114}

The complete solution then is both displacements added into one equation:

\[ x_1 = C_{11} \sin (\omega_1 t + \Phi_1) + C_{12} \sin (\omega_2 t + \Phi_2) \]

\[ \lambda_1 C_{21} \sin (\omega_1 t + \Phi_1) + \lambda_2 C_{22} \sin (\omega_2 t + \Phi_2) \] \tag{1.115}

and

\[ x_2 = C_{21} \sin (\omega_1 t + \Phi_1) + C_{22} \sin (\omega_2 t + \Phi_2) \] \tag{1.116}

The values of \( C_{11}, C_{12}, C_{21}, C_{22}, \Phi_1, \) and \( \Phi_2 \) depend on the initial conditions and the mode shapes, \( \lambda_1 \) and \( \lambda_2 \).

**Ex. 1.7** is a 2DOF free and undamped CLO system.

**1.10.2 Free Damped 2DOF System.** Adding damping complicates the equations considerably, but the procedure remains the same. In this case, it is easier to use linear algebra. Using the same model as before (see **Fig. 1.16**), but adding viscous dampers (\( c_1 \) and \( c_2 \neq 0 \)), the equations of motion for the independent DOFs are determined for mass 1 as
Example 1.7—2DOF free and undamped CLO system.

Given that \( m_1 = 2 \text{ lbm}, m_2 = 2 \text{ lbm}, k_1 = 2 \text{ lbf/ft}, k_2 = 2 \text{ lbf/ft} \), determine the equations of motion for mass 1 and mass 2:

\[
\begin{align*}
    m_1 \ddot{x}_1 &= -k_1 x_1 + k_2 (x_2 - x_1), \quad (1.94) \\
m_2 \ddot{x}_2 &= -k_2 (x_2 - x_1), \quad (1.95)
\end{align*}
\]

Rearranging gives:

\[
\begin{align*}
    m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 &= 0 \quad (1.117) \\
m_2 \ddot{x}_2 - k_2 x_1 + k_2 x_2 &= 0 \quad (1.118)
\end{align*}
\]

Assume a solution of the form \( x = C \sin(\omega t + \phi) \):

\[
\begin{align*}
    x_1 &= C_1 \sin(\omega t + \phi), \quad (1.96) \\
x_2 &= C_2 \sin(\omega t + \phi), \quad (1.97)
\end{align*}
\]

which when differentiated twice gives

\[
\begin{align*}
    \ddot{x}_1 &= -\omega^2 C_1 \sin(\omega t + \phi), \quad (1.98) \\
    \ddot{x}_2 &= -\omega^2 C_2 \sin(\omega t + \phi). \quad (1.99)
\end{align*}
\]

Substituting into the equations of motion gives:

\[
\begin{align*}
    -m_1 \omega^2 C_1 \sin(\omega t + \phi) + (k_1 + k_2) C_1 \sin(\omega t + \phi) - k_2 C_2 \sin(\omega t + \phi) &= 0 \quad (1.119) \\
    -m_2 \omega^2 C_2 \sin(\omega t + \phi) + k_2 C_1 \sin(\omega t + \phi) - k_2 C_1 \sin(\omega t + \phi) &= 0 \quad (1.120)
\end{align*}
\]

which simplifies to

\[
\begin{pmatrix}
    -m_1 \omega^2 + k_1 + k_2 & -k_2 \\
    -k_2 & -m_2 \omega^2 + k_2
\end{pmatrix}
\begin{pmatrix}
    C_1 \\
    C_2
\end{pmatrix}
= 0. \quad (1.121)
\]

The only nontrivial way these equations can equal zero is if the determinant of the coefficients is equal to zero. Therefore:

\[
\begin{pmatrix}
    -m_1 \omega^2 + k_1 + k_2 & -k_2 \\
    -k_2 & -m_2 \omega^2 + k_2
\end{pmatrix} = 0 \quad (1.102)
\]

The solution is:

\[
\begin{pmatrix}
    -m_1 \omega^2 + k_1 + k_2 \\
    -k_2
\end{pmatrix} \begin{pmatrix}
    -m_2 \omega^2 + k_2
\end{pmatrix} = 0 \quad (1.122)
\]

which gives the characteristic equation

\[
m_1 m_2 \omega^4 - \left[ m_1 k_2 + m_2 (k_1 + k_2) \right] \omega^2 + k_2 k_2 = 0. \quad (1.103)
\]

which is a quadratic function in terms of \( \omega^2 \). Therefore, \( \omega^2 \) is equal to:

\[
\omega^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{m_1 k_2 + m_2 (k_1 + k_2) \pm \sqrt{(m_1 k_2 + m_2 (k_1 + k_2))^2 - 4m_1 m_2 k_2 k_2}}{2m_1 m_2}. \quad (1.104)
\]

which gives two solutions, which means two natural frequencies. In this example, the frequencies are \( \omega_1 = 9.178 \text{ Hz} \) and \( \omega_2 = 3.506 \text{ Hz} \).

The shape functions are:

\[
\begin{align*}
    \lambda_1 &= \frac{k_2}{k_1 + k_2 - \omega_1^2 m_1} = -1.618 \quad (1.107) \\
    \lambda_2 &= \frac{k_2}{k_1 + k_2 - \omega_2^2 m_1} = 0.618 \quad (1.111)
\end{align*}
\]

To determine the specific solutions, the boundary conditions must be applied. At time equal zero, they are \( x_1 = 1 \text{ ft}, x_2 = 0 \text{ ft} \), \( v_1 = 0 \text{ ft/s}, \) and \( v_2 = 0 \text{ ft/s} \). Applying these to the solution to the equations of motion gives:

\[
\begin{align*}
    x_1 &= C_{11} \sin(\omega t + \phi_1) + C_{12} \sin(\omega t + \phi_2), \quad (1.123) \\
    x_2 &= C_{21} \sin(\omega t + \phi_1) + C_{22} \sin(\omega t + \phi_2), \quad (1.124) \\
    v_1 &= \omega C_{11} \sin(\omega t + \phi_1) + \omega C_{12} \sin(\omega t + \phi_2), \quad (1.125) \\
    v_2 &= \omega C_{21} \sin(\omega t + \phi_1) + \omega C_{22} \sin(\omega t + \phi_2). \quad (1.126)
\end{align*}
\]

which gives four unknowns for four independent equations. The simultaneous solution to the above set of equations is

\[
\begin{align*}
    C_{21} &= -0.447 \text{ ft}, \quad C_{22} = 0.447 \text{ ft}, \quad \phi_1 = 1.571 \text{ radians}, \quad \phi_2 = 1.571 \text{ radians}; \quad \text{and further solutions give } C_{11} = 0.724 \text{ ft} \quad \text{and} \quad C_{12} = 0.276 \text{ ft}.
\end{align*}
\]

The solution to the equations of motion is:

\[
\begin{align*}
    x_1 &= 0.724 \sin\left[2\pi(9.178)t + 1.571\right] + 0.276 \sin\left[2\pi(3.506)t + 1.571\right] \quad \text{and} \\
    x_2 &= -0.447 \sin\left[2\pi(9.178)t + 1.571\right] + 0.447 \sin\left[2\pi(3.506)t + 1.571\right],
\end{align*}
\]

where \( 2\pi \) is for conversion to radians.
\[ m_1 \ddot{x}_1 = -k_1 x_1 + k_2 (x_2 - x_1) - c_1 \dot{x}_1 + c_2 (\dot{x}_2 - \dot{x}_1) \]................................. (1.127)

and for mass 2 as
\[ m_2 \ddot{x}_2 = -k_2 (x_2 - x_1) - c_2 (\dot{x}_2 - \dot{x}_1). \]................................. (1.128)

This can be written in matrix form as
\[
\begin{bmatrix}
  m_1 & 0 \\
  0 & m_2
\end{bmatrix}
\begin{bmatrix}
  \ddot{x}_1 \\
  \ddot{x}_2
\end{bmatrix}
+ \begin{bmatrix}
  c_1 + c_2 & -c_2 \\
  -c_2 & c_2
\end{bmatrix}
\begin{bmatrix}
  \dot{x}_1 \\
  \dot{x}_2
\end{bmatrix}
+ \begin{bmatrix}
  k_1 + k_2 & -k_2 \\
  -k_2 & k_2
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix}
= \begin{bmatrix}
  0 \\
  0
\end{bmatrix}
\]................................. (1.129)

or as
\[ M \ddot{X} + C \dot{X} + K X = 0, \]....................................................... (1.130)

where \( M = \) the mass matrix, \( C = \) the damping matrix, \( K = \) the stiffness matrix, \( \ddot{X} \) = the acceleration vector, \( \dot{X} \) = the velocity vector, and \( X \) = the displacement vector, which are given as:
\[ M = \begin{bmatrix}
  m_1 & 0 \\
  0 & m_2
\end{bmatrix}, \].......................................................... (1.131)

\[ C = \begin{bmatrix}
  c_1 + c_2 & -c_2 \\
  -c_2 & c_2
\end{bmatrix}, \].......................................................... (1.132)

\[ K = \begin{bmatrix}
  k_1 + k_2 & -k_2 \\
  -k_2 & k_2
\end{bmatrix}, \].......................................................... (1.133)

\[ \ddot{X} = \begin{bmatrix}
  \ddot{x}_1 \\
  \ddot{x}_2
\end{bmatrix}, \].......................................................... (1.134)

\[ \dot{X} = \begin{bmatrix}
  \dot{x}_1 \\
  \dot{x}_2
\end{bmatrix}, \].......................................................... (1.135)

and
\[ X = \begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix}. \].......................................................... (1.136)

If the solution is assumed to be of the form
\[ x = C e^{\omega t}, \]....................................................... (1.137)
then

\[ \dot{x} = \omega C e^{\omega t} \]  ............................................................... (1.138)

and

\[ \ddot{x} = \omega^2 C e^{\omega t} \]  .............................................................. (1.139)

Substituting back into the equation of motion, the result in matrix form is:

\[ \omega^2 M C e^{\omega t} + \omega C C e^{\omega t} + K C e^{\omega t} = 0, \]  .................................................. (1.140a)

which can be rewritten as

\[ (\omega^2 M + \omega C + K) C e^{\omega t} = 0. \]  .................................................. (1.140b)

Because time is always positive and a nontrivial solution is desired, the only way this equation is true is if the determinate of the coefficient of \( C e^{\omega t} \) is zero; that is, if

\[ \left| \omega^2 M + \omega C + K \right| = 0. \]  .................................................. (1.141)

The determinate of Eq. 1.141 is a fourth-order polynomial in terms of \( a \), which means that there are four roots. These roots can be:
Example 1.8—2DOF free and damped CLO system.

Given \( m_1 = 2 \text{ lbm}, m_2 = 2 \text{ lbm}, c_1 = 0.1 \text{ lbf-ft/sec}, c_2 = 0.1 \text{ lbf-ft/sec}, k_1 = 2 \text{ lbf/ft}, \) and \( k_2 = 2 \text{ lbf/ft} \), determine the equations of motion for mass 1 and mass 2:

\[
\begin{align*}
    m_1 \ddot{x}_1 &= -k_1 x_1 + k_2 (x_2 - x_1) - c_1 \dot{x}_1 + c_2 (\dot{x}_2 - \dot{x}_1) \quad (1.127) \\
    m_2 \ddot{x}_2 &= -k_2 (x_2 - x_1) - c_2 (\dot{x}_2 - \dot{x}_1) \quad (1.128)
\end{align*}
\]

Rearranging gives:

\[
\begin{align*}
    m_1 \ddot{x}_1 + (c_1 + c_2) \dot{x}_1 - c_2 \ddot{x}_2 + (k_1 + k_2) x_1 - k_2 x_2 &= 0 \quad (1.142) \\
    m_2 \ddot{x}_2 - c_1 \dot{x}_1 + c_2 \ddot{x}_2 - k_3 x_1 + k_2 x_2 &= 0 \quad (1.143)
\end{align*}
\]

Using linear algebra can simplify the notation. The mass matrix is

\[
\mathbf{M} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \quad (1.131)
\]

the damping matrix is

\[
\mathbf{C} = \begin{pmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{pmatrix} \quad (1.132)
\]

the stiffness matrix is

\[
\mathbf{K} = \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix} \quad (1.133)
\]

the acceleration vector is

\[
\ddot{\mathbf{x}} = \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} \quad (1.134)
\]

the velocity vector is

\[
\dot{\mathbf{x}} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} \quad (1.135)
\]

and, finally, the displacement vector is

\[
\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (1.136)
\]

which simplifies the basic equation of motion to

\[
\mathbf{M} \ddot{\mathbf{x}} + \mathbf{C} \dot{\mathbf{x}} + \mathbf{K} \mathbf{x} = \mathbf{0} \quad (1.130)
\]

Assume a solution of the form \( \mathbf{x} = \mathbf{C} e^{\omega t} \):

\[
\begin{align*}
    x_1 &= C_{11} e^{\omega t} \quad (1.144) \\
    x_2 &= C_{21} e^{\omega t} \quad (1.145)
\end{align*}
\]

which, when differentiated once gives

\[
\begin{align*}
    \dot{x}_1 &= \omega C_{11} e^{\omega t} \quad (1.146) \\
    \dot{x}_2 &= \omega C_{21} e^{\omega t} \quad (1.147)
\end{align*}
\]

and when differentiated twice gives

\[
\begin{align*}
    \ddot{x}_1 &= \omega^2 C_{11} e^{\omega t} \quad (1.148) \\
    \ddot{x}_2 &= \omega^2 C_{21} e^{\omega t} \quad (1.149)
\end{align*}
\]

Substituting into the equations of motion gives:

\[
(\omega^2 \mathbf{M} + \omega \mathbf{C} + \mathbf{K}) \mathbf{C} e^{\omega t} = \mathbf{0}, \quad (1.140\text{b})
\]

where

\[
\mathbf{C} = \begin{pmatrix} C_{11} \\ C_{21} \end{pmatrix} \quad (1.150)
\]

1. Four real and negative roots.
2. Two sets of complex conjugates with negative real parts.
3. Two real and negative roots and one set of complex conjugates.
As in the previous example, the only nontrivial way this equation can equal zero is if the determinant of the coefficients is equal to zero. Therefore, completely spelled out, it becomes:

\[
\begin{bmatrix}
m_1 \omega^2 + (c_1 + c_2) \omega + (k_1 + k_2) & -c_2 \omega - k_2 \\
-c_2 \omega - k_2 & m_2 \omega^2 + c_2 \omega + k_2
\end{bmatrix} = 0.
\] .......................... (1.151)

The solution gives the characteristic equation:

\[
m_1 m_2 \omega^4 + \left[ m_1 c_2 + m_2 \left( c_1 + c_2 \right) \right] \omega^3 + \left[ m_1 k_2 + m_2 \left( k_1 + k_2 \right) + c_2 c_2 \right] \omega^2 + \left( c_2 k_2 + c_2 k_2 \right) \omega + k_2 k_2 = 0,
\] .......................... (1.152)

which is a fourth-order polynomial with four roots. The roots of this equation are:

\[
\begin{align*}
\omega &= -2.106 - 8.933i \\
\omega &= -2.106 + 8.933i \\
\omega &= -0.307 - 3.492i \\
\omega &= -0.307 + 3.492i.
\end{align*}
\]

This is a case where the roots are pairs of complex conjugates. This gives a decaying exponential oscillating type of solution. If one takes the absolute value of \(\omega\), the results are:

\[
\omega = 9.178 \\
\omega = 3.506.
\]

Or two natural frequencies, as expected.

There are four shape functions:

\[
\lambda = \begin{bmatrix}
\alpha_1 c_2 + k_2 \\
\alpha_2 m_2 + \alpha_1 \left( c_1 + c_2 \right) + \left( k_1 + k_2 \right) \\
\alpha_2 c_2 + k_2 \\
\alpha_2 m_2 + \alpha_1 \left( c_1 + c_2 \right) + \left( k_1 + k_2 \right)
\end{bmatrix} = \begin{bmatrix}
\alpha_1 m_2 + \alpha_0 c_2 + k_2 \\
\alpha_2 m_2 + \alpha_0 c_2 + k_2 \\
\alpha_2 c_2 + k_2 \\
\alpha_2 m_2 + \alpha_0 c_2 + k_2
\end{bmatrix}
\]

To determine the specific solutions, the boundary conditions must be applied. At time equal zero, they are:

\(x_1 = 1\) ft, \(x_2 = 0\) ft, \(v_1 = 0\) ft/sec, \(v_2 = 0\) ft/sec.

Applied to the solution to the equations of motion, the displacements are

\[
x_1 = C_1 e^{\omega_1 t} + C_2 e^{\omega_2 t} + C_3 e^{\omega_3 t} + C_4 e^{\omega_4 t}
\]

and the initial velocities are

\[
v_1 = \omega_1 C_1 e^{\omega_1 t} + \omega_2 C_2 e^{\omega_2 t} + \omega_3 C_3 e^{\omega_3 t} + \omega_4 C_4 e^{\omega_4 t}
\]

These are eight unknowns for four equations. Further constraints include:

\[
C_1 = \lambda_1 C_{22}, \quad C_2 = \lambda_1 C_{22}, \quad C_3 = \lambda_i C_{23}, \quad C_4 = \lambda_i C_{24}.
\]

which gives eight unknowns for eight independent equations. The simultaneous solutions to the above set of equations are:

\[
C_1 = 0.362 - 0.085i, \quad C_2 = 0.362 + 0.085i, \quad C_3 = 0.138 + 0.012i, \quad C_4 = 0.138 - 0.012i,
\]

\[
C_{22} = -0.224 + 0.053i, \quad C_{23} = -0.224 - 0.053i, \quad C_{24} = 0.224 + 0.020i, \quad C_{25} = 0.224 - 0.020i.
\]

The solutions to the equations of motion are:

\[
x_1 = (0.362 - 0.085i) e^{2 \left(-2.106 - 8.933i\right)t} + (0.362 + 0.085i) e^{2 \left(-2.106 + 8.933i\right)t} + (0.138 - 0.012i) e^{2 \left(-0.307 - 3.492i\right)t} + (0.138 + 0.012i) e^{2 \left(-0.307 + 3.492i\right)t} 
\]

\[
x_2 = (-0.224 + 0.053i) e^{2 \left(-2.106 - 8.933i\right)t} + (-0.224 - 0.053i) e^{2 \left(-2.106 + 8.933i\right)t} + (0.224 - 0.020i) e^{2 \left(-0.307 - 3.492i\right)t} + (0.224 + 0.020i) e^{2 \left(-0.307 + 3.492i\right)t}
\]

Example 1.8—2DOF free and damped CLO system (continued).
then the motions will be exponentially decaying oscillations for both DOFs. This is similar to
the underdamped case for an SDOF system. Finally, for case number three, either condition
can occur.

**Ex. 1.8** is a 2DOF free and underdamped CLO system.

More information on linear algebra can be found in Chap. 2 of this section of the Handbook.

### 1.10.3 Forced Damped 2DOF System

Adding forcing complicates the equations considerably yet again. The procedure is the same, however, and it is a matter of keeping the mathematics straight. Many texts are available to delve more deeply into this subject. For more information, please refer to Refs. 11 through 19.

### 1.10.4 Multiple-DOF Systems

The previous discussion of 2DOF systems points out how to handle any DOF system. The last example used matrix notation to define the system for the solution process. Multiple-DOF systems are solved similarly with the primary difference being the degree of the defining matrices is greater, as is the degree of difficulty in solving the system. The matrix will have the same number of rows and columns as the degree of freedom. There are other methods (e.g., finite-element modeling) that can be used to tackle the complexity of multiple-DOF systems.

### 1.11 Continuous Systems

If one continues to add DOFs, the limit at an infinite DOF defines a continuous system. The result becomes a PDE. The following is a brief description of the separation of variables method for solving a PDE.

**Fig. 1.17** shows a freebody diagram for axial and torsional systems. The axial system equations will be used to determine the solution of the equations of motion. Eq. 1.162 is the axial equation of motion:

$$m \frac{\partial^2 u}{\partial t^2} = A_c E \frac{\partial^2 u}{\partial x^2} - c \frac{\partial u}{\partial t} - mg_c, \quad \text{................................. (1.162)}$$

where $m \frac{\partial^2 u}{\partial t^2}$ = the inertial force, $A_c E \frac{\partial^2 u}{\partial x^2}$ = the rate of strain change, $mg_c$ = the static weight of the element, and $c \frac{\partial u}{\partial t}$ = the force from viscous damping. This PDE, Eq. 1.162, can be solved using the separation of variables method. This is shown as:

$$U(x, t) = X(x)T(t), \quad \text{................................. (1.164)}$$

The following solution assumption is made concerning the time function:

$$T(t) = e^{i\omega t}, \quad \text{................................. (1.165)}$$

This equation is substituted back into the assumed solution, which then is appropriately differentiated and substituted back into the equation of motion. The equation becomes

$$A_c E \frac{\partial^2 X(x)}{\partial x^2} e^{i\omega t} + (m\omega^2 - c\omega)X(x)e^{i\omega t} = 0, \quad \text{................................. (1.166)}$$

which is of the form
The standard solution of this equation is:

\[ X(x) = C_1 e^{-i\phi x} + C_2 e^{i\phi x} \] ................................. (1.168)

The constants of integration, \( C_1 \) and \( C_2 \), are determined by the initial and boundary conditions, and \( \phi \) is a collection of the constants and is given by:

\[ \phi = \sqrt{\frac{m\omega^2 - ci\omega}{A_e E}} \] ................................. (1.169)

Therefore, the total solution is:

\[ U(x, t) = (C_1 e^{-i\phi x} + C_2 e^{i\phi x})e^{i\omega t} \] ................................. (1.170)

The solution to the torsional equation of motion is derived similarly to the axial equation, with the substitution of the appropriate variables and noting that there is no initial strain from gravity. The variables \( u, m, A_e, E, c, \omega, v_s \), and \( \phi \) are replaced by \( \theta, I, J, G, c_\theta, \omega_\theta, v_\theta \), and \( \eta \), respectively. The torsional equation of motion is:

\[ I \frac{\partial^2 \theta}{\partial t^2} = JG \frac{\partial^2 \theta}{\partial x^2} - c_\theta \frac{\partial \theta}{\partial t} \] ................................. (1.163)

This gives the solution as:
\[ \theta(x, t) = \left[(C_1 e^{-i\eta x} + C_2 e^{i\eta x})e^{i\omega \theta t}\right]. \]                          \hfill (1.171)

Constants \( C_1 \) and \( C_2 \) are based on the initial and boundary conditions, and \( \eta \) is a collection of the constants and is given by:

\[ \eta = \sqrt{\frac{\omega_0^2}{v_\theta^2} - \frac{i c_\theta \rho_\theta}{J G}}, \]                          \hfill (1.172)

where

\[ v_\theta = \sqrt{\frac{J G}{m_\theta}}, \]                          \hfill (1.173)

and

\[ m_\theta = \frac{\rho A_c}{8} (d_2^2 - d_1^2). \]                          \hfill (1.174)

### 1.12 Wave Reflection From Various Geometric Boundaries

As discussed early in the chapter, wave propagation is the movement of a distinct group of waves through some material in response to an external force.

A key point in wave-propagation studies is how waves interact with geometric discontinuities. What happens as a wave meets a fixed or free boundary condition? Also, what happens to a wave as it encounters a geometrical area change or a change in material properties?

There are two limiting boundary conditions for wave propagation: a fixed (pinned) end (zero displacement) and a free end (zero stress). A fixed end is a boundary condition in which there is zero displacement. According to wave theory, during a wave encounter with a fixed end, the stress at the fixed end doubles during the passage of the wave. A reflection of a stress wave will simply bounce back with the same sign. A compression wave will reflect as a compression wave and a tension wave will reflect as a tension wave. At a fixed end, because the displacement is zero, the particle velocity will be zero. The wave particle velocity amplitude is inverted during a reflection from a fixed end.

A free end is defined as a boundary condition free to move. The stress at the free end is always zero. The effects on stress and particle velocity caused by a free end are opposite of the effects on stress on a fixed end. A compression wave encountering a free end reflects as a tension wave, and a tension wave reflects as a compression wave. The wave particle velocity values double during an encounter with a free end and reflect with the same sign.

As a wave encounters a change in cross-sectional area, some of the wave is reflected and some is transmitted (refracted). The amplitudes and sign of the waves depend on the relative change in cross-sectional area. The equation that describes the effect on the incident force, \( F_i \), of a cross-sectional-area, density, or modulus-of-elasticity change for the transmitted wave is

\[ F_t = \frac{\sqrt{\frac{E_2 \rho_2}{E_1 \rho_1}}}{\sqrt{\frac{E_2 \rho_2}{E_1 \rho_1}} \frac{A_c^2}{A_{c2}} + 1} F_i, \]                          \hfill (1.175)
and for the reflected wave is

\[
F_r = \frac{\sqrt{\frac{2}{E_2 \rho_2} A_{c2}} - 1}{\sqrt{\frac{2}{E_1 \rho_1} A_{c1}}} \frac{F_i}{\sqrt{\frac{2}{E_2 \rho_2} A_{c2}} + 1} \tag{1.176}
\]

If an incident wave encounters a junction where the relative change in cross-sectional area is greater than 1 (a smaller area to a larger area), most of the wave will transmit through the junction. Some of the wave will reflect from the junction and will keep the same sign. For example, a compression wave will transmit through the junction and keep going as a somewhat-diminished compression wave. The part of the wave that is reflected is still a compression wave, but its amplitude is less than that of the wave that transmitted through the junction.

On the other hand, if an incident wave encounters a junction where the relative change in cross-sectional area is less than 1 (a larger area to a smaller area), most of the wave will reflect off the junction, but some of it will transmit through the junction and will keep the same sign. For example, a compression wave will transmit through the junction and keep going as a diminished compression wave. The reflected part of the wave is a tension wave whose absolute amplitude is greater than that of the compression wave that is transmitted through the junction.

As with most drillstrings, there are many geometric discontinuities (changes in cross-sectional area) that will cause part of the wave to refract and part to reflect. For example, drill collars to heavyweight drillpipe to drillpipe all are geometric discontinuities. Sometimes, too, there are material discontinuities—changes in material density or modulus of elasticity—that cause refractions and reflections. A third possible type of discontinuity is when there are different endpoints. For example, if the pipe is stuck, one end can be modeled as stuck. If the pipe is hanging freely, such as with casing running, then the end is free.

More-detailed information on wave propagation can be found in Refs. 20 through 25.

**Nomenclature**

\[
\begin{align*}
\alpha & = \text{acceleration, } L/t^2, \text{ ft/sec}^2 \\
A & = \text{slope, dimensionless} \\
A_c & = \text{cross-sectional area, } L^2, \text{ in.}^2 \\
c & = \text{axial damping coefficient, } mL/t, \text{ lbf-ft/sec} \\
c_{\text{crit}} & = \text{critical damping coefficient, dimensionless} \\
c_{\theta} & = \text{torsional damping coefficient, } mL/t, \text{ lbf-sec/rad} \\
C & = \text{constant of integration, various} \\
C & = \text{damping matrix, } mL/t, \text{ lbf-ft/sec} \\
d_i & = \text{inner diameter, } L, \text{ in.} \\
d_o & = \text{outer diameter, } L, \text{ in.} \\
E & = \text{modulus of elasticity, } m/Lt^2, \text{ psia} \\
E(t) & = \text{excitation function} \\
f & = \text{cyclic frequency, } 1/t, \text{ cycle/sec} \\
f(t) & = \text{function of } t \\
f'(t) & = \text{first differential function} \\
F & = \text{axial force, } mL/t^2, \text{ lbf} \\
F_d & = \text{damping force, } mL/t^2, \text{ lbf} \\
F_f & = \text{friction force, } mL/t^2, \text{ lbf} \\
F_h & = \text{hysteretic force, } mL/t^2, \text{ lbf}
\end{align*}
\]
\( F_i \) = incident force, mL/t², lbf
\( F_n \) = normal force, mL/t², lbf
\( F_r \) = reflected force, mL/t², lbf
\( F_t \) = transmitted force, mL/t², lbf
\( F_0 \) = initial force, mL/t², lbf
\( g_c \) = gravitational constant, L/t², 32.174 ft/sec²
\( g(t) \) = function of \( t \)
\( g'(t) \) = first derivative of function of \( g(t) \)
\( G \) = shear modulus, m/Lt², psia
\( h \) = hysteretic factor, dimensionless
\( i \) = imaginary operator
\( I \) = second moment of inertia, L⁴, in.⁴
\( j \) = iteration index
\( J \) = polar moment, L³, in.³
\( k \) = spring constant, m/t², lbf/in.
\( k_{eq} \) = equivalent spring constant, m/t², lbf/in.
\( k_w \) = wave number, 1/L, 1/ft
\( K \) = stiffness matrix, m/t², lbf/in.
\( L \) = total length, L, ft
\( m \) = mass, m, lbm
\( m_{θ} \) = mass polar moment of inertia, mL, lbf-sec²
\( M \) = mass matrix, m, lbm
\( n \) = exponent
\( P \) = generic value
\( Q \) = generic value
\( r \) = generic constant
\( r_j \) = iterated generic constant
\( R(t) \) = response function
\( t \) = time, seconds
\( T \) = torque, mL²/t², lbf-ft
\( T(t) \) = displacement function in terms of time, \( t \)
\( u \) = displacement, L, in.
\( U(x,t) \) = continuous displacement function, L, in.
\( v \) = velocity, L/t, ft/sec
\( v_g \) = group speed, L/t, ft/sec
\( v_s \) = sonic velocity, L/t, ft/sec
\( v_w \) = phase velocity, L/t, ft/sec
\( v_0 \) = initial velocity, L/t, ft/sec
\( v_θ \) = torsional sonic velocity, L/t, ft/sec
\( x \) = displacement, L, in.
\( x_h \) = homogeneous displacement, L, in.
\( x_p \) = particular displacement, L, in.
\( x_0 \) = initial displacement, L, in.
\( \dot{x} \) = first derivative with respect to time of displacement (velocity), L/t, ft/sec
\( \dot{x}_p \) = particular velocity, L/t, ft/sec
\( \ddot{x} \) = second derivative with respect to time of displacement (acceleration), L/t², ft/sec²
\( \ddot{x}_p = \) particular acceleration, \( \text{L/t}^2, \text{ft/sec}^2 \)

\( X(x) = \) displacement function in terms of location \( x \)

\( \mathbf{X} = \) displacement vector

\( \dot{\mathbf{X}} = \) velocity vector, \( \text{L/t, ft/sec} \)

\( \ddot{\mathbf{X}} = \) acceleration vector, \( \text{L/t}^2, \text{ft/sec}^2 \)

\( y = \) dependent variable, various

\( y_i = \) distance at point \( i \), L, in.

\( Y = \) limit value, various

\( z = \) arbitrary constant, various

\( \Delta l = \) change in length, L, in.

\( \Delta t = \) change in time, t, seconds

\( \Delta y = \) change in dependent variable, various

\( \alpha = \) amplitude, various

\( \varepsilon = \) strain, L/L, in./in.

\( \eta = \) convenient coefficient, 1/L, 1/ft

\( \theta = \) twist, rad

\( \dot{\theta} = \) second derivative with respect to time of twist (acceleration) rad/sec\(^2\)

\( \lambda = \) mode shape, dimensionless

\( \lambda_{ao} = \) wavelength, L, in.

\( \mu = \) friction factor, dimensionless

\( \zeta = \) critical damping ratio, dimensionless

\( \rho = \) density, m/L\(^3\), lbm/in.\(^3\)

\( \sigma = \) stress, m/Lt\(^2\), psia

\( \Phi = \) phase angle, rad

\( \omega = \) frequency, 1/t, Hz

\( \omega_d = \) damped natural frequency, 1/t, Hz

\( \omega_f = \) forcing frequency, 1/t, Hz

\( \omega_n = \) natural frequency, 1/t, Hz

\( \omega_\theta = \) twist natural frequency, 1/t, Hz

\( \varphi = \) convenient coefficient, 1/L, 1/ft

\( \tau = \) period, t, seconds

References


Suggested Reading


Appendix—Glossary of Vibration Theory Terms

The **period** of a vibration is the time taken for a motion to repeat.

**Frequency** is the number of repeats per unit of time. This also is called a cyclic frequency. An angular frequency (sometimes called circular frequency) ($\omega$) is measured in radians per unit time. Angular frequency is the product of $2\pi$ and the cyclic frequency.\(^7\)

The **amplitude** of a frequency is the difference between the maximum or minimum values of a sinusoidally varying quantity and its mean. A wave’s crest is its maximum amplitude, and the trough is its minimum amplitude.

The **wave period** is the length of time it takes a wave to travel from crest to crest. The wave frequency is the inverse of the wave period.

The **wavelength** is the distance covered by one wave period.

The **phase angle** (abbreviated to “phase”) of a wave is the fraction of the distance the wave has traveled relative to an arbitrary reference. The points at which the various waves cross the average amplitude line do not coincide unless the phase is zero. The phase is negative if the crossing occurs before the reference wave crossing, and positive if it occurs after the reference wave crossing.

**Phase velocity** is the velocity of a given point, as defined by the phase angle on a sinusoidal wave. When the phase velocity and the frequency remain constant, it is a nondispersing wave. If the phase velocity and the frequency are not constant, the wave disperses.

The **wave number** is the ratio of the wavelength to $2\pi$.\(^{10}\) If this ratio is a real number, the wave will be a nondispersing wave (i.e., it will have a constant phase velocity for all its components and will retain its shape). If the wave number is complex, then the wave will be a dispersing wave (i.e., its components will have different phase velocities and it will not retain its shape and will disperse over time). All waves encountered in macroscopic reality are dispersing to some extent.

The **group speed** is the response of a wave’s group of sinusoidal components, and is different from phase velocity. Group speed is important; the interaction of all the wave components in terms of wave number, amplitude, and frequency causes the appearance of a carrier wave of some wave number, amplitude, and frequency. This carrier wave is modulated by group waves that propagate at group speeds. In a nondispersing wave, the group speeds are the same as the...
phase velocity of the wave components. In a dispersing wave, however, these group speeds can be different. If the group speeds are greater than the phase velocities, it will appear that a carrier wave originates at the back of the wave, propagates forward until it reaches the front of the wave, vanishes, and reappears at the back of the wave. If the group speeds are less than the phase velocities, it will appear that a carrier wave originates at the front of the wave, propagates backward until it reaches the back of the wave, vanishes, and reappears at the front of the wave. This can be visualized by thinking of a caterpillar’s motion. The caterpillar’s ripples appear to move backward, yet the caterpillar moves forward. This is one reason the analysis of dispersing waves can be very complicated.

The **natural frequency** of a system is the frequency at which the system vibrates when free of any friction or forcing functions.

A **damped natural frequency** is a natural frequency with friction.

**Damping** is the dissipation of energy with time or distance. Several forms of damping are discussed in this chapter.

A **harmonic** is a frequency that is an integer multiple of a given frequency.

A **mode** is a particular harmonic frequency.

The **mode shape** is a pattern of the system assumed during a harmonic.

A **frequency spectrum** is a band of frequencies.

A **forcing function** is an external force that is acting on a vibrating system. This has the effect of modifying the vibrations that are experienced in the system, sometimes significantly. For example, in a rotating-drillstring system, the drill bit will impart a forcing vibration to the lower end of the string.

A **critical frequency** is a frequency at which amplitudes become unbounded. This occurs when the frequency of the system matches one of the natural frequencies.

The **critical rotary speed** is a rotary speed that coincides with one of the critical frequencies of the system.

**Resonance** occurs when the frequency of the forcing function is equal to one of the natural frequencies of the system.

A **stable** system returns to its former position after the removal of the force that disturbed the system. In pipe work, stability is involved with buckling, bending, and straightness of the pipe.

An **incident wave** is a wave going into a geometric or material discontinuity.

A **transmitted wave** is the portion of a wave that passes through a geometric or material discontinuity.

A **reflected wave** is the portion of a wave that does not pass through a geometric or material discontinuity.

**Impedance** is the ratio of a force-like quantity to a velocity-like quantity when the arguments of the real or imaginary parts of the quantities linearly increase with time. The reciprocal of impedance is **mobility**. If the mechanical impedance of two materials and their geometries are equal, then the maximum amount of energy will be transmitted at their intersection.

The **beating phenomenon** occurs when two harmonic waves of slightly different frequencies are impressed on a body. They are a periodic variation in vibration at a frequency that is the difference between to two frequencies.

**Self-excited** vibrations occur when the forcing function is a function of the displacement, velocity, or acceleration of the system mass. If the energy gained from the self-excited vibrations is more than the system can absorb, the system will self-destruct.

A **standing wave** is a wave that appears to be standing still. There is no discernible beginning or end in time to the wave.

A **transient wave** is a wave that has a definite beginning and eventually dies out.

**Strain energy** is the energy gained by a mass as it is deformed.